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Multiscaling and localized instabilities in fracture, fragmentation, and growth processes

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Abstract. It is shown that localized instabilities can be an origin of log-normal and power-law statistical distributions in fracture, fragmentation and island growth processes. Results of laboratory experiments and numerical simulations performed by different authors are used to demonstrate the applicability of this approach.

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1 Introduction

Fragmentation and island growth processes usually exhibit power or log-normal distributions. However, the nature of these distributions can be very different in different kinds of fragmentation and growth processes (see, for instance [1–15] and references therein). Moreover, experiments and numerical simulations show that in the same fragmentation process both power-law and log-normal distributions can take place, depending on conditions (amount of impact energy, for instance). To find a *dynamical* mechanism leading to the broad statistical distributions encountered in these processes seems to be actual and interesting problem. The authors of a recent paper [4] suggested to relate the statistical distribution of fragment sizes of brittle solids to the statistical distribution of the length of the branches which appear as result of instability of main cracks propagation. They based this hypothesis on an observation. Namely, they observed *log-normal* distribution for the lengths of microbranches which appear in a brittle plastic as a result of a straight crack propagation instability. This attractive hypothesis raises additional questions. What is the mechanism generating the log-normal distribution of the branch lengths themselves? Can this mechanism generate the power-law distribution as well?

As shown in [4] appearance and growth of the branches can be considered as a *dynamical* process closely related to the localized instability of the main straight crack propagation. A specific feature of this growth process is the sudden interruption of the growth of individual branches at some stage. Another growth process exhibiting analogous dynamical properties was recently discovered in a numerical simulation of islands growth in molecular beam epitaxy [16]. In this case we also have fast (exponential) growth of island sizes related to localized instability with interruption of the fast growth at some stage. Using data obtained in this numerical simulation we shall show below that in this case we also have a log-normal distribution of the island sizes. It should be noted that a log-normal distribution of the islands size is also observed in numerous experiments for island solid surface roughening [9–15].

Thus one can see from the experiment [4] and the numerical simulation [16] that dynamical growth processes related to localized instabilities with interruption on some stage lead to broad (log-normal) distributions. In the present paper we suggest a simple explanation of this phenomenon that relates to the appearance of the broad statistical distributions (the power-law as well) to localized (dynamical) instabilities.

2 Fracture

In reference [4] results of fracture experiments performed on PMMA (poly-methyl-methacrylate) are reported. These experiments indicate that the transition from a single-crack to a multicrack state is the result of a dynamical instability. Namely, when the velocity of the crack exceeds a critical velocity v_c a single-crack state no longer exists. Instead, a crack sprouts small microscopic side branches whose dynamics are interrelated with those of the main crack. These side branches propagate for a short distance and die, the main crack velocity develops oscillations, and a nontrivial structure is formed on the fracture surface. From a dynamical point of view these

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side branches can be considered as metastable states with some life-times τ.

The simplest equation describing distribution of the lifetimes of the metastable states (the branches) along the axis x, corresponding to the main straight crack propagation line, is a Langevin-type equation

$$\frac{\mathrm{d}\tau}{\mathrm{d}x} = f(\tau, x) + \eta(x) \quad (1)$$

where $f(\tau, x)$ is a function on τ and x and $\eta(x)$ is a white noise with:

$$\langle \eta(x)\eta(x') \rangle = 2\gamma\delta(x - x').$$

For a homogeneous situation $f(\tau, x)$ does not depend on x. The Fokker-Plank equation corresponding to equation (1) is

$$\frac{\partial P(\tau, x)}{\partial x} = -\frac{\partial f(\tau)P(\tau, x)}{\partial \tau} + \gamma \frac{\partial^2 P(\tau, x)}{\partial \tau^2} \cdot \quad (2)$$

The space-homogeneous (*i.e.* independent on x) solution of this equation is

$$P(\tau) = N \exp\left[\frac{1}{\gamma}\int_0^\tau f(\tau')\mathrm{d}\tau'\right] \quad (3)$$

where N is a normalization constant.

Expanding $f(\tau)$ into a Maclaurin series in a vicinity of $\tau = 0$

$$f(\tau) = a_0 + a_1\tau + a_2\tau^2 + ... \quad (4)$$

one can study the distribution for small enough τ. If $a_0 < 0$ one obtains from (3)

$$P(\tau) \sim \mathrm{e}^{-|a_0|\tau/\gamma} \quad (5)$$

i.e. an exponential distribution with a local maximum at $\tau = 0$. If $a_0 > 0$ one should take into account the next term in the expansion (4)

$$f(\tau) \simeq a_0 + a_1\tau \quad (6)$$

and one obtains from (3) for $a_1 < 0$

$$P(\tau) \sim \mathrm{e}^{-(\tau-\overline{\tau})^2/\sigma^2} \quad (7)$$

where $\sigma_0^2 = 2\gamma/|a_1|$ and $\overline{\tau} = a_0/|a_1|$. This is a Gaussian distribution.

Let us now estimate the probability distribution of the branch lengths. In the linear stage of the instability [17] one can estimate the time of growth of the velocity fluctuations as

$$v(t) \simeq v_0 \exp(\sigma t) \quad (8)$$

where v_0 is an initial velocity of the branch propagation. The length of the branch is

$$l(\tau) = l_0 + \int_0^\tau v(t)\ \mathrm{d}t \quad (9)$$

where τ is lifetime of the corresponding metastable state and l_0 is the initial value of the length. Substituting (8) into (9) we obtain

$$l(\tau) \simeq l_0 + \frac{v_0}{\sigma}(\mathrm{e}^{\sigma\tau} - 1). \quad (10)$$

Since we know the statistical distribution of τ we can find the statistical distribution of $l(\tau)$ using (10). If we take the exponential distribution of the life-times given by (5), then using (10) we obtain

$$P(l) \sim (l - l_{\min})^\alpha \quad (11)$$

where $\alpha = 1 + |a_0|/\gamma\sigma$, and $l_{\min} = l_0 - v_0/\sigma$. This is a power-law distribution with some shift (for $l \gg l_{\min}$ this is the usual power-law distribution). If we now take the Gaussian distribution of life-times (7) we obtain a log-normal distribution of the branch lengths

$$P(l) \sim \frac{1}{l - l_{\min}} \exp\left\{-\frac{1}{\sigma_2^2}\left[\ln\frac{l - l_{\min}}{l_{\mathrm{m}} - l_{\min}}\right]^2\right\} \quad (12)$$

where

$$l_{\mathrm{m}} = l_0 + \frac{v_0}{\sigma}(\mathrm{e}^{\sigma\overline{\tau}} - 1)$$

and $\sigma_2 = \sigma_1\sigma$. This is a log-normal distribution with some minimal value $l = l_{\min}$. It should be noted that the authors of reference [4] fitted their experimental data by log-normal distribution just of such type.

3 Fragmentation

The authors of [4] suggest to relate the distribution of the branch lengths to the distribution of fragment sizes of crushed or fractured objects (see Introduction). Using this assumption and the results of the previous section, we can now understand why the two types of the probability distribution: log-normal and power-law, are usually observed in the numerous experiments and numerical simulations of the fragmentation processes. For instance, in paper [1] results of an experiment with long, thin glass rods are presented. At low falling height (about 1 m drop), the fragment-size distribution can be described by the log-normal law while at falling heights greater than about 4.5 m, a power-law distribution gives a good fit to the data. In another paper [5] impact fragmentation of an ideal brittle crystal is numerically simulated using Newton equation of motion and deterministic rule of breakage (rather than stochastic modeling) with a full consideration of a viscoelastic dynamics of material. The result of this simulation also show the two types of fragment distributions: a power-law distribution at early stage (when the additional energy introduced by the impact is relatively high) and a log-normal distribution in the late stage of the fragmentation process (when most of the additional impact-energy is already dissipated). Natural (tectonic) observations also exhibit these two types of statistical distribution [2].

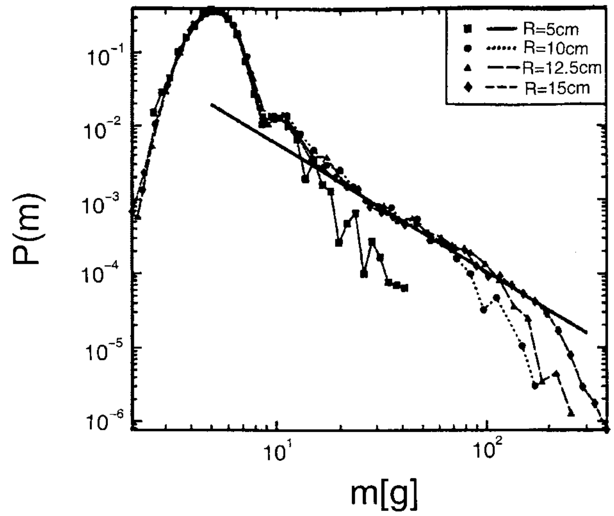

Fig. 1. Probability distribution of the fragment mass for fragmentation of colliding disks (numerical simulation [6]).

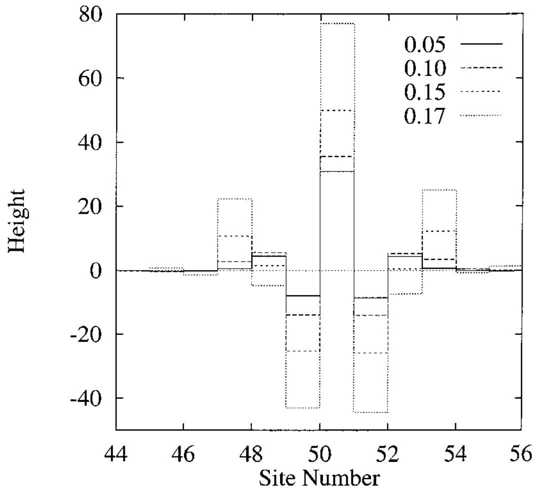

Fig. 2. Interface profiles of development of the instability induced by the presence of a pillar of initial height $h_0 = 25$ at times $t = 0.05, 0.1, 0.15$ and 0.17 [16].

It should be noted that the exponential probability distribution of the life-times similar to the small life-time representation (5) can be also obtained for large life-times in the ergodic case. Indeed, if there exists the ergodic limit

$$\lim_{\tau\to\infty} \frac{\int_0^\tau f(\tau')\mathrm{d}\tau'}{\tau} = \langle f \rangle \tag{13}$$

then for large life-times one can use the approximation

$$\int_0^\tau f(\tau')\mathrm{d}\tau' \simeq \langle f \rangle \tau \tag{14}$$

and obtain for this case from (3)

$$P(\tau) \sim \mathrm{e}^{-|\langle f \rangle|\tau/\gamma} \tag{15}$$

for $\langle f \rangle < 0$. For the ergodic situation the linear stage of the dynamical approach to the metastable states can be sufficiently long and the corresponding distribution of the branch length will be similar to (11) (*i.e.* a power-law), only the exponent α should be replaced by $\alpha = 1 + |\langle f \rangle|/\gamma\sigma$. In this case one can simultaneously observe both, a log-normal distribution related to the small sizes and a power-law distribution related to the large sizes of the fragments. Indeed, in a recent paper [6] fragmentation of colliding discs was studied numerically using a cell model of brittle solids. Figure 1 (taken from [6]) shows the mass distribution of the fragments obtained for an initial velocity of the disks $v = 50$ m/s for different system sizes ($R = 5$ cm, 10 cm, 12.5 cm, 15 cm). Since log-log scales are used in this figure the log-normal distribution has the shape of a parabola, while the power-law distribution corresponds to a straight line. One can see in this figure that for small fragments we have a log-normal fragment mass distribution while for comparatively large fragments the distribution can be fitted by a power-law.

4 Islands growth

It is quite common to see the *log-normal* distribution used to fit the distribution of islands size in experiments on solid surface roughening (see some recent papers [9–15]). Analogous situation takes also place in experiments on corrosion pits [11]. Recent discovery [16] of a relation between multiscaling and instabilities of discretized growth equations (and related atomistic models) to isolated pillars (or grooves) growth allows to relate this experimentally observed log-normality to a generic instability of the surface growth processes.

The Lai-Das Sarma equation for the height $h(\mathbf{r}, t)$ at the point $\mathbf{r}$ of the flat surface at time t

$$\frac{\partial h}{\partial t} = -\nu\nabla^4 h + \lambda\nabla^2(\nabla h)^2 + \eta(\mathbf{r}, t), \tag{16}$$

with a Gaussian noise $\eta(\mathbf{r}, t)$, is suggested in [19] to describe the molecular beam epitaxy (here ν and λ are some constants). A numerical simulation performed in [16] for a discretized version of this equation shows that there exists a generic instability in which isolated pillars (or grooves) on an otherwise flat interface grow in time when their height (or depth) exceeds a critical value h_c. Figure 2 (taken from [16]) shows the development of the instability induced in the 1D discretized LD equation (for $\lambda = 1$) by the presence of a pillar of initial (dimensionless) height $h_0 = 25$. In the initial stage, the height is zero everywhere except at the 50th site where the height is h_0. Interface profiles are shown at dimensionless time $t = 0.05, 0.1, 0.15$ and 0.17. If we calculate the development in time of the height of the main and the secondary pillars (grooves) using this figure we obtain that for $t \leq 0.15$ the height $h(t)$ grows exponentially with time (see Fig. 3)

$$h(t) = h_0 \mathrm{e}^{\sigma t}. \tag{17}$$

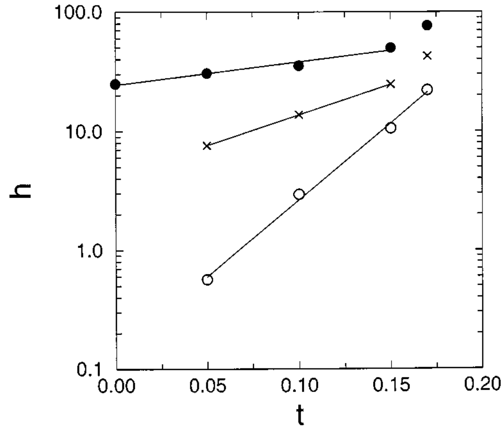

Fig. 3. Height (depth) of the pillars (grooves) against t as shown in Figure 2. Closed circles correspond to main pillar, open circles correspond to secondary pillars, crosses correspond to secondary grooves.

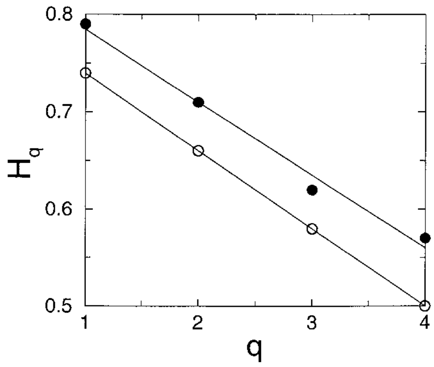

Fig. 4. Generalized roughness exponents H_q against q for two types of the controlled instabilities (data taken from [16]). Closed circles correspond to the control using modified deposition rule and open circles correspond to the control using terms with high power of gradients of height variable. The straight lines are drawn to indicate linear dependence on q (log-normal representation (22)).

This is typical for the *linear* stage of growth of the real instabilities of nonlinear equations [17]. For $t > 0.15$ there is a strong acceleration of the growth and it is expected that just at this stage introduction of the high order nonlinear terms in the LD equation should interrupt the anomalous rapid growth (see [16] and [18] for detail argumentation of the necessity of such high order nonlinearities in the LD equations and so-called controlled instabilities). Hence one can introduce a lifetime, τ, of the "activity" (the exponential growth) of the pillars. The pillars appear on the flat surface (or line in 1D case) in a random way and one can apply the approach developed in Section 2 to obtain the log-normal (and power-law) distribution of the pillars size.

Let us now show how this log-normal distribution is related to the multiscaling observed in [16] for the controlled instabilities.

The generalized roughness exponents, H_q, are introduced through the various moments of the height differences

$$c_q(r) = \frac{1}{N}\sum_{i=1}^{N} |h(x_i) - h(x_i + r)|^q \sim r^{qH_q} \tag{18}$$

where N is the number of points over which the average is taken and the limit $r \to 0$ is considered. For standard partition $r \sim 1/N$. The authors of [20] assumed that, when evaluating (18), r and N may be related in a way different from $r \sim 1/N$. Namely, $N \sim r^{-D_0}$ (D_0 could be considered here as fractal dimension of the support of the process). Using this assumption and the definition of the generalized dimensions D_q [20]

$$Z_q(r) = \sum_{i=1}^{N} p_i^q(r) \sim r^{(q-1)D_q} \tag{19}$$

where

$$p_i(r) = \frac{|h(x_i) - h(x_i + r)|}{\sum_{j=1}^{N} |h(x_j) - h(x_j + r)|}$$

a relation between the generalized dimensions and the generalized roughness exponents

$$H_q = H_1 + \frac{(q-1)(D_q - D_0)}{q} \tag{20}$$

can be obtained.

For the log-normal distribution

$$D_q = D_0 - aq \tag{21}$$

where a is a constant. Then substituting (21) into (20) we obtain

$$H_q = (H_1 + a) - aq \tag{22}$$

i.e. for the log-normal distribution the generalized roughness exponent is a linear function of the order q (*cf.* non-island growth [21]). In Figure 4 we show H_q obtained in numerical simulations performed in [16] for two types of controlled instabilities for the atomistic models corresponding to the LD equation [22]. In the model of the first type, the instability is controlled by introducing terms with high power of gradients of height variable, whereas the second type consists of atomic models in which instability is controlled by modifying the deposition rule. One can see from Figure 4 that in both these cases H_q is approximately linear function of q, that corresponds to the log-normal distribution.

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